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An Algorithm for the Constrained Longest Common Subsequence and Substring Problem

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Subsequences and Substrings

-Let ∑ be an alphabet and **S** a string over ∑. A subsequence of a string **S** is obtained by deleting zero or more letters from **S**.

If **S** = “ACGTU”, then “ATU” is a subsequence of **S**.

-A substring of a string **S** is a subsequence of S consists of consecutive letters in **S**.

If **S** = “ACGTU”, then “CGT” is a substring of **S,** “ATU” is not a substring of **S**,

-Every substring of **S** is also a subsequence of **S**.

-The empty string is a subsequence and a substring of any string.

The Longest Common Subsequence Problem for Two Strings

-The longest common subsequence problem for two strings X and Y is to find a longest string, denoted LCSSeq(X, Y), which is a subsequence of both X and Y.

-Obviously, the set of LCSSeq(X, Y) and the set LCSSeq(Y, X) are the same.

$|LCSSeq(X, Y)| = |LCSSeq(Y, X)|$.

The Longest Common Substring Problem for Two Strings

-The longest common substring problem for two strings X and Y is to find a longest string, denoted LCSStr(X, Y), which is a substring of both X and Y.

-Obviously, the set of LCSStr(X, Y) and the set LCSStr(Y, X) are the same.

|LCSStr(X, Y)| = |LCSStr(Y, X)|.

The Longest Common Subsequence and Substring Problem for Two Strings

-In [1], Li, Deka, and Deka introduced the longest common subsequence and substring problem for two strings X and Y which is to find a longest string, denoted LCSSeqStr(X, Y), that is a subsequence of X and a substring Y.

-[1] R. Li, J. Deka, and K. Deka, An algorithm for the longest common subsequence and substring problem, Journal of Math and Informatics 25 (2023) 77-81.

The Longest Common Subsequence and Substring Problem for Two Strings

-In general, the set of LCSSeqStr(X, Y) and the set LCSSeqStr(Y, X) are not the same. |LCSSeqStr(X, Y)| $\neq$ |LCSSeqStr(Y, X)|.

-In [1], Li, Deka, and Deka designed an algorithm for LCSSeqStr(X, Y). The time and space complexities of the algorithm are O(|X| |Y|), where |X| and |Y| are the lengths of strings X and Y, respectively.

Examples

- Suppose **X** = “GAAAACCCT” and **Y** = “GACACACT”.

-“AC” is a longest common substring for **X** (resp. **Y**) and **Y (**resp. **X)**. Thus |LCSStr(**X**, **Y**)| = |LCSStr(**Y**, **X**)| = 2.

-“ACT” is a longest common subsequence and substring for **X** and **Y**. Thus |LCSSeqStr(**X**, **Y**)| = 3.

-“ACCCT” is a longest common subsequence and substring for **Y** and **X**. Thus |LCSSeqStr(**Y**, **X**)| = 5.

-“GAAACT” is a longest common subsequence for **X** (resp. **Y**) and **Y** (resp. **X**). Thus LCSSeq(**X**, **Y**)| = |LCSSeq(**Y**, **X**)| = 6.

The Three Problems

-$|LCSStr(X, Y)| = |LCSStr(Y, X)| \leq |LCSSeqStr(X, Y)| \leq$

$|LCSSeq(X, Y)| = |LCSSeq(Y, X)|$.

-$|LCSStr(X, Y)| = |LCSStr(Y, X)| \leq |LCSSeqStr(Y, X)| \leq$

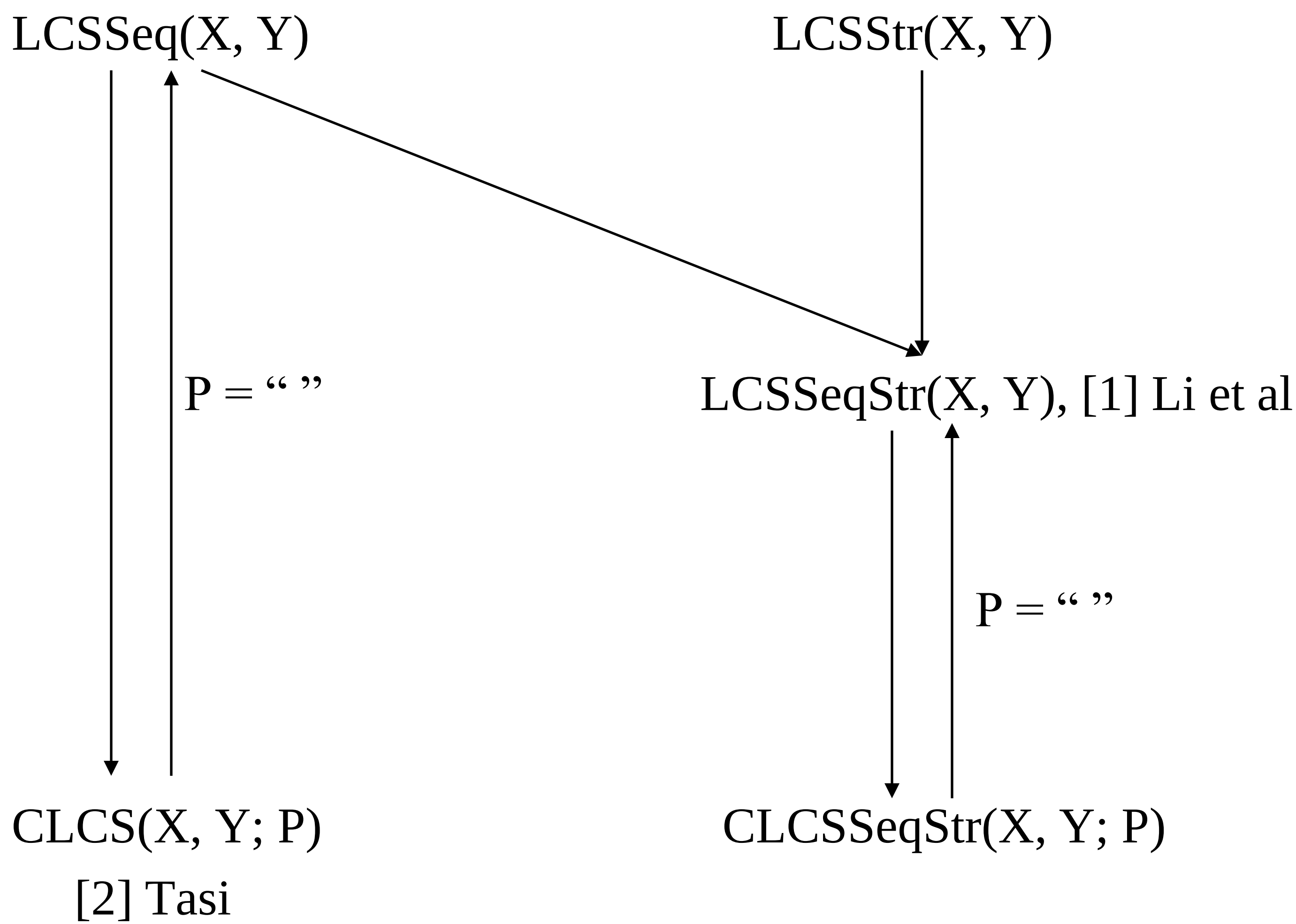
$|LCSSeq(X, Y)| = |LCSSeq(Y, X)|$.

-$|LCSStr(X, Y)| = |LCSStr(Y, X)| \leq \min\{|LCSSeqStr(X, Y)|, |LCSSeqStr(Y, X)|\}$

$\leq \max\{|LCSSeqStr(X, Y)|, |LCSSeqStr(Y, X)|\}$

$\leq |LCSSeq(X, Y)| = |LCSSeq(Y, X)|$.

The Problems



The Constrained Longest Common Subsequence Problem for Two Strings

-Tsai [2] extended the longest common subsequence problem for two strings to the constrained longest common subsequence problem for two strings and a constrained string P.

-[2] Y. T. Tsai, The constrained longest common subsequence problem,

Information Processing Letters 88 (2003) 173-176.

The Constrained Longest Common Subsequence Problem for Two Strings

-For two strings X, Y, and a constrained string P, the constrained longest common subsequence problem for two strings X and Y with respect to P is to find a longest string Z: = CLCSSeq(X, Y; P) such that Z is a subsequence of both X and Y and P is a subsequence of Z.

-Clearly, if P is an empty string, then CLCSSeq(X, Y; P) = LCSSeq(X, Y).

The Constrained Longest Common Subsequence Problem for Two Strings

-“Such a problem could arise in computing the homology of two biological sequences which have a specific or putative structure in common” quoted from [2].

-Tsai [2] designed an $O(|X^2 |Y|^2 |P|)$ time algorithm for CLCSSeq(X, Y; P).

The Constrained Longest Common Subsequence and Substring Problem for Two Strings

-Motivated by Tasi’s work, we introduced the constrained longest common subsequence and substring problem for two strings and a constrained string.

The Constrained Longest Common Subsequence and Substring Problem for Two Strings

-For two strings X, Y, and a constrained string P, the constrained longest common subsequence and substring problem for two strings X and Y with respect to P, is to find a longest string Z: = CLCSSeqStr(X, Y; P) such that Z is a subsequence of X, a substring of Y, and P is a subsequence of Z.

-Clearly, if P is an empty string, then CLCSSeqStr(X, Y; P) = LCSSeqStr(X, Y) in [1].

The Constrained Longest Common Subsequence and Substring Problem for Two Strings

-Suppose **X** = “GAAAACCCT”, **Y** = “GACACACT”, **P** = “AC”.

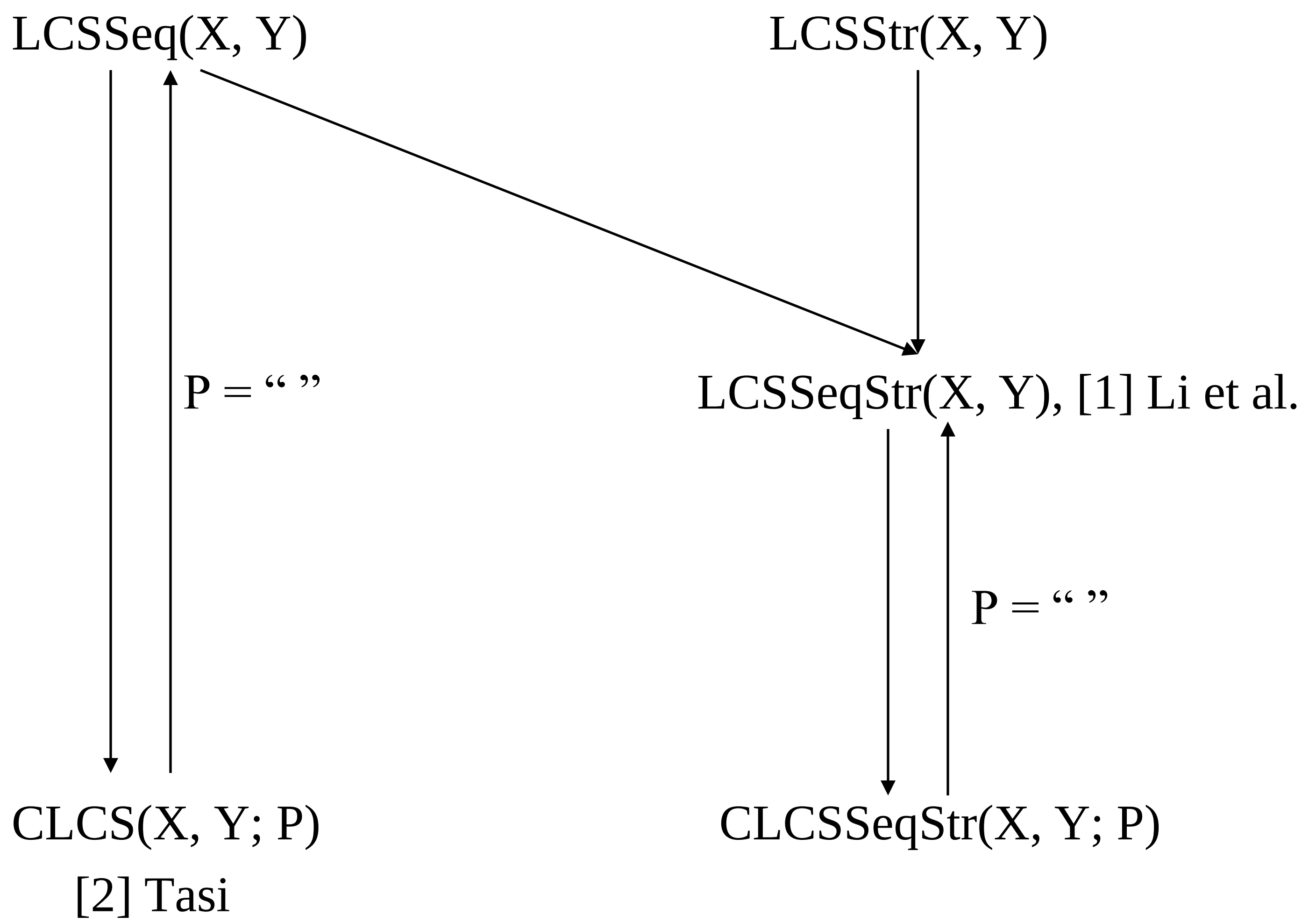
-“ACT” is a constrained longest common subsequence and substring for **X** and **Y**.

Thus |CLCSSeqStr(**X**, **Y; P**)| = 3.

-“GAAACT” is a constrained longest common subsequence for **Y** (resp. **X**) and **X** (resp. **Y**). Thus |CLCSSeq(**X**, **Y; P**)| = |LCSSeq(**Y**, **X; P**)| = 6.

-In general, |CLCSSeqStr(**X**, **Y; P**)| ≤ |CLCSSeq(**X**, **Y; P**)|.

The Problems



The Constrained Longest Common Subsequence and Substring Problem for Two Strings

-Using dynamic programing (DP), we designed an algorithm for finding CLCSSeqStr(X, Y; P). Both time complexity and space complexity of our algorithm are O(|X| |Y| |P|).

The Algorithm

-Let $S = s_1\ s_2\ ...\ s_r$ be a string over an alphabet $\sum$, The r prefixes of

S are defined as $S_1 = s_{1,}\ S_2 = s_1 s_2,\ S_3 = s_1\ s_2\ s_3, \ldots,$ and $S_r = s_1\ s_2\ ...\ s_r$.

S_0 is defined as an empty string.

-The r suffixes of S are defined as $T_1 = s_1\ s_2\ ...\ s_r$, $T_2 = s_2\ s_3\ ...\ s_r$, ..., $T_{r-1} = s_{r-1}s_r$,

and $T_r = s_r$,

The Algorithm

-Let $X = x_1\ x_2\ ...\ x_m$, $Y = y_1\ y_2\ ...\ y_n$, and $P = p_1\ p_2\ ...\ p_r$. Define Z[i, j, k] as a string

satisfying the following conditions.

(1) it is a subsequence of $X_i = x_1\ x_2\ ...\ x_{i,}$

(2) it is a suffix of $Y_j = y_1\ y_2\ ...\ y_{j,}$

(3) it has $P_k = p_1\ p_2\ ...\ p_k$ as a subsequence,

(4) under (1), (2) and (3), its length is as large as possible,

where $1 \le i \le m$, $1 \le j \le n$, and $1 \le k \le r$.

The Algorithm

-We will use a 3-dimensional array $M[m + 1][n + 1][r + 1]$ to store $|Z[i, j, k]|$.

Namely, $M[i][j][k] = |Z[i, j, k]|$, where $0 \leq i \leq m$, $0 \leq j \leq n$, $0 \leq k \leq r$.

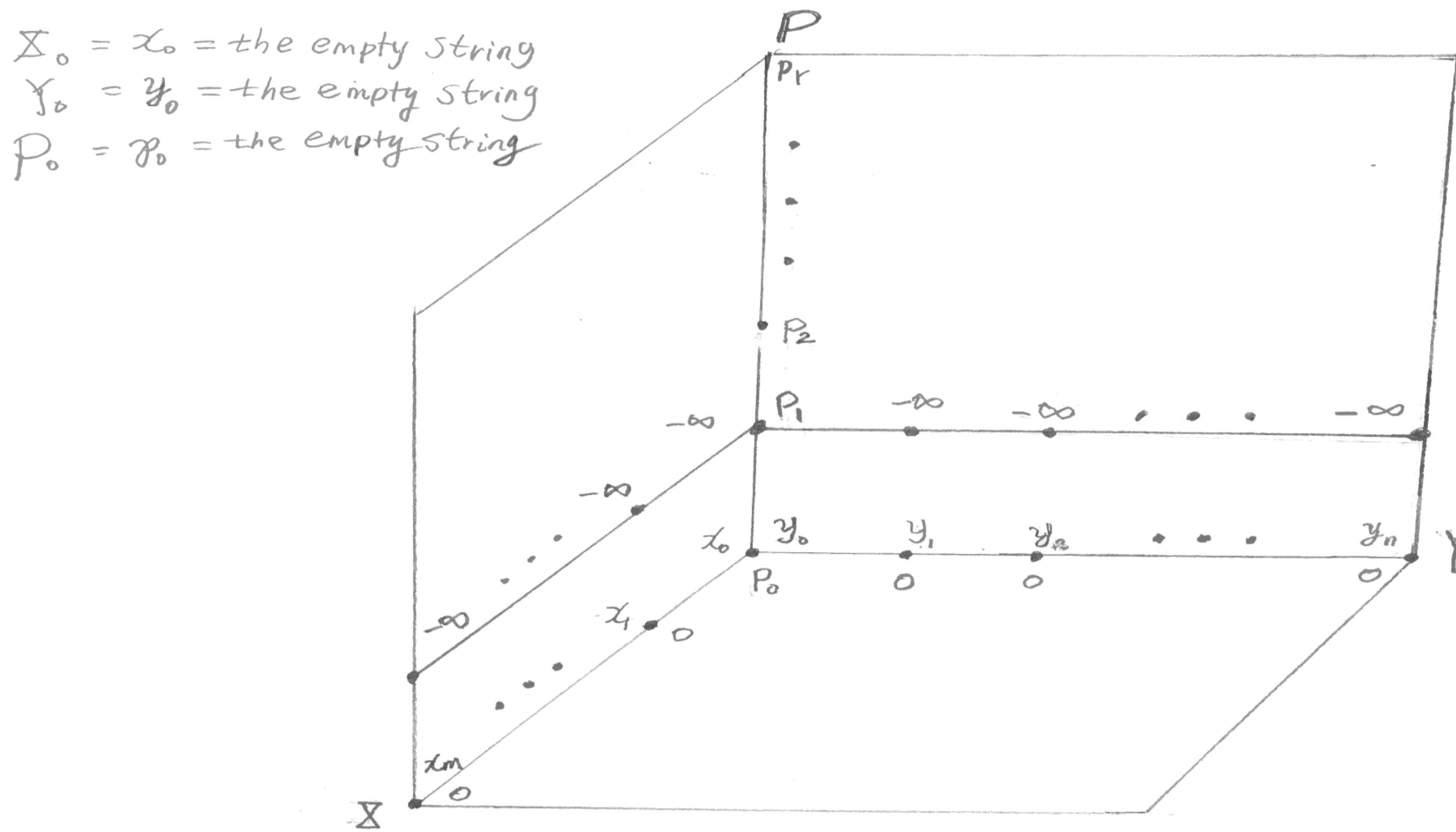
-We will recursively fill in the cells in M.

-Firstly, we fill in the boundary cells in array M.

The Algorithm

-Filling in the boundary cells.

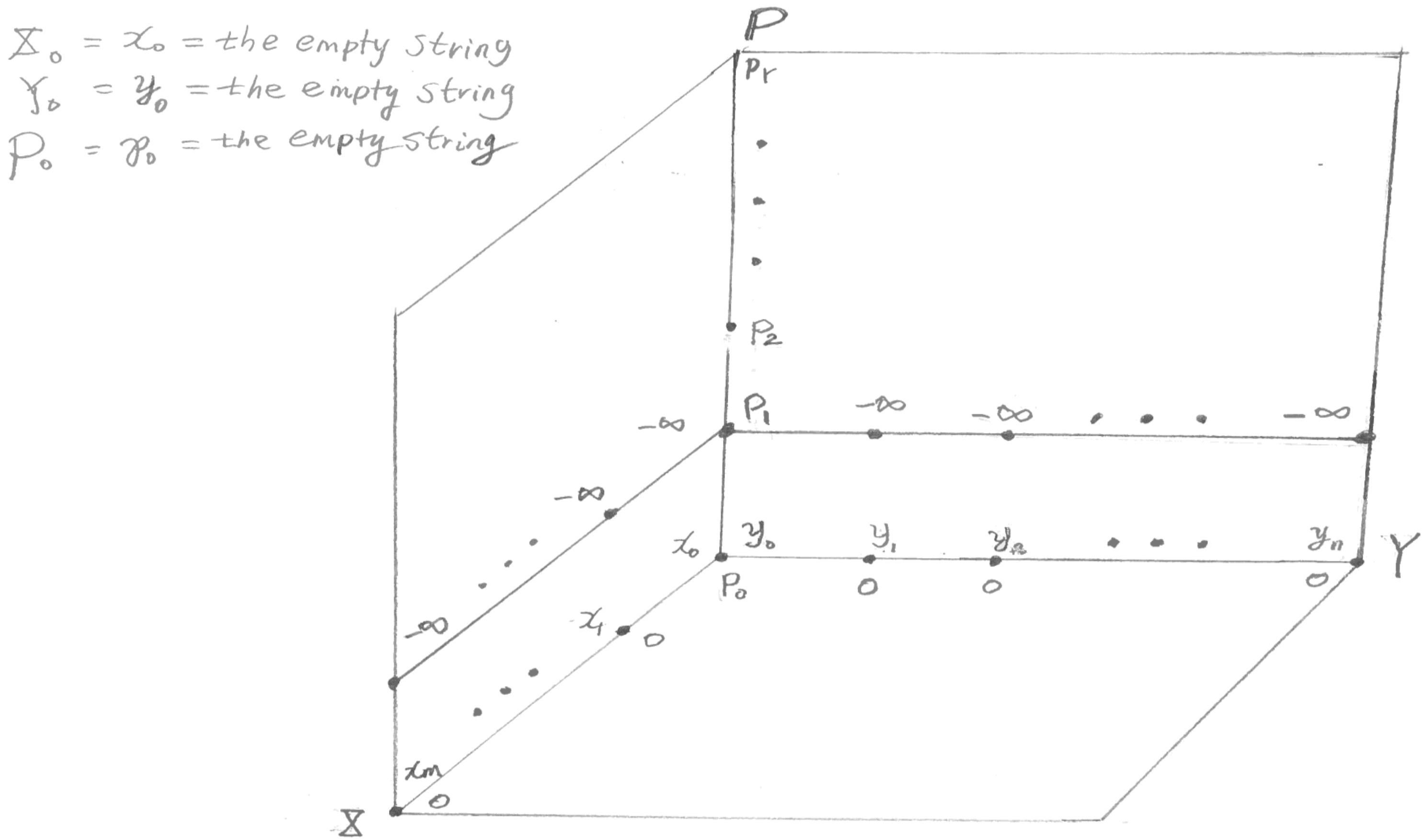
[1] If $i = 0$ and $k = 0$ the length of a string which is a subsequence of X_i, a suffix of Y_j, and has P_k as a subsequence is zero. Thus $M[0][j][0] = 0$, where $0 \le j \le n$.



The Algorithm

-Filling in the boundary cells.

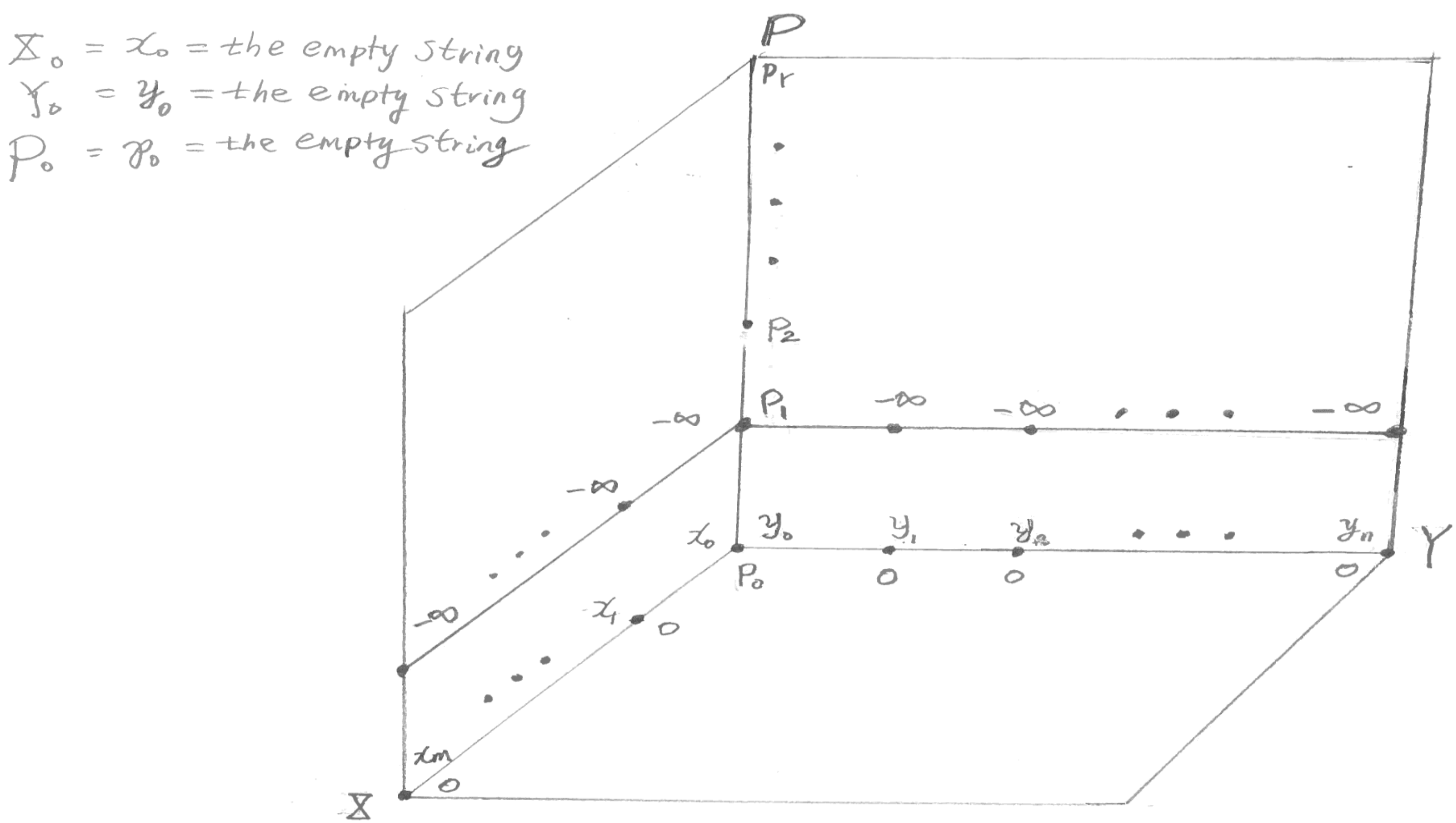
[2] If j = 0 and k = 0, the length of a string which is a subsequence of X_i, a suffix of Y_j, and has P_k as a subsequence is zero. Thus M[i][0][0] = 0, where $0 \leq i \leq m$.



The Algorithm

-Filling in the boundary cells.

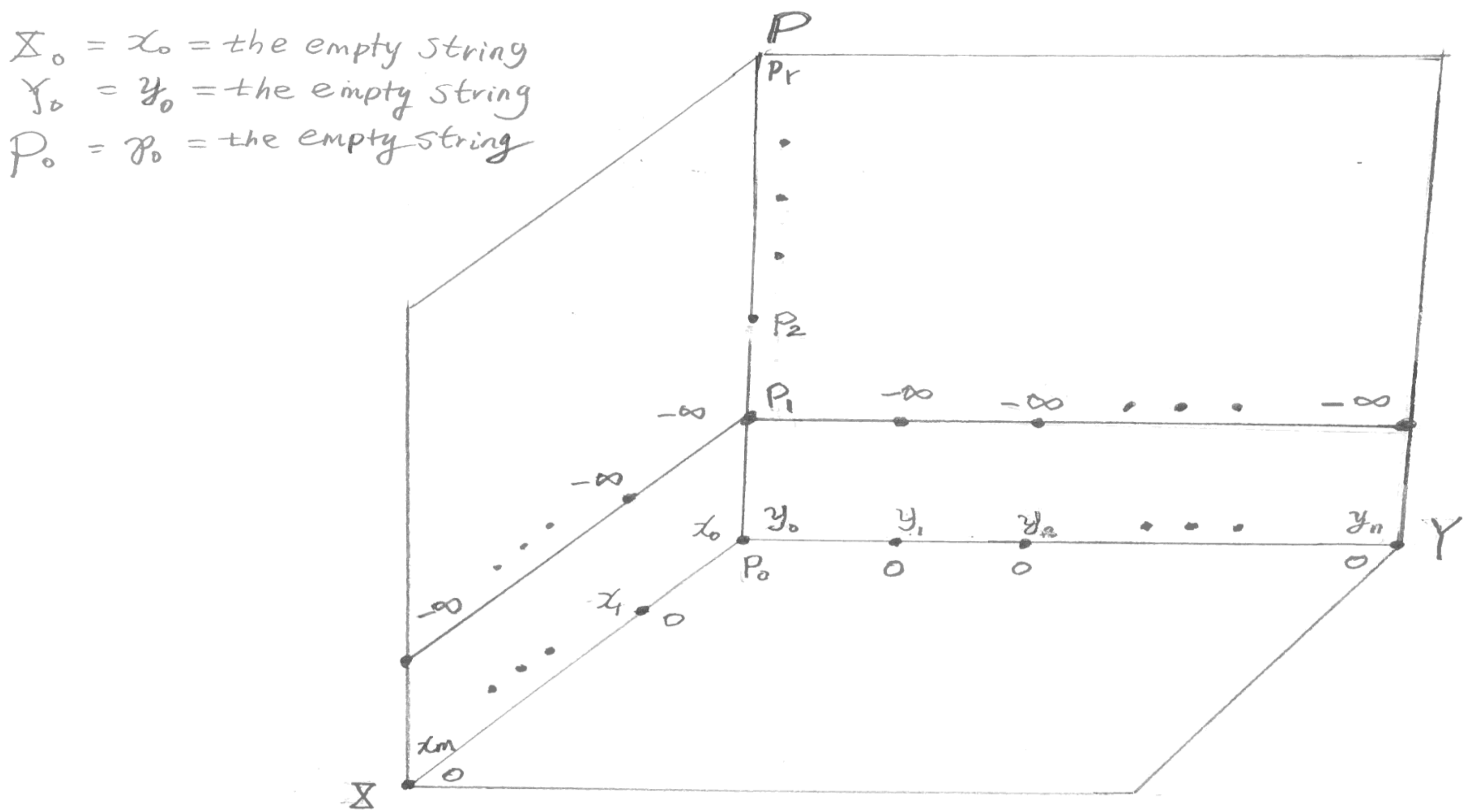
[3] If $i = 0$ and $k \geq 1$, there is not a string which is a subsequence of X_i, a suffix of Y_j, and has P_k as a subsequence. Thus $M[0][j][k] = -\infty$, where $0 \leq j \leq n$ and $1 \leq k \leq r$.



The Algorithm

-Filling in the boundary cells.

[4] If $j = 0$ and $k \geq 1$, there is not a string which is a subsequence of X_i, a suffix of Y_j, and has P_k as a subsequence. Thus $M[i][0][k] = -\infty$, where $0 \leq i \leq m$ and $1 \leq k \leq r$.

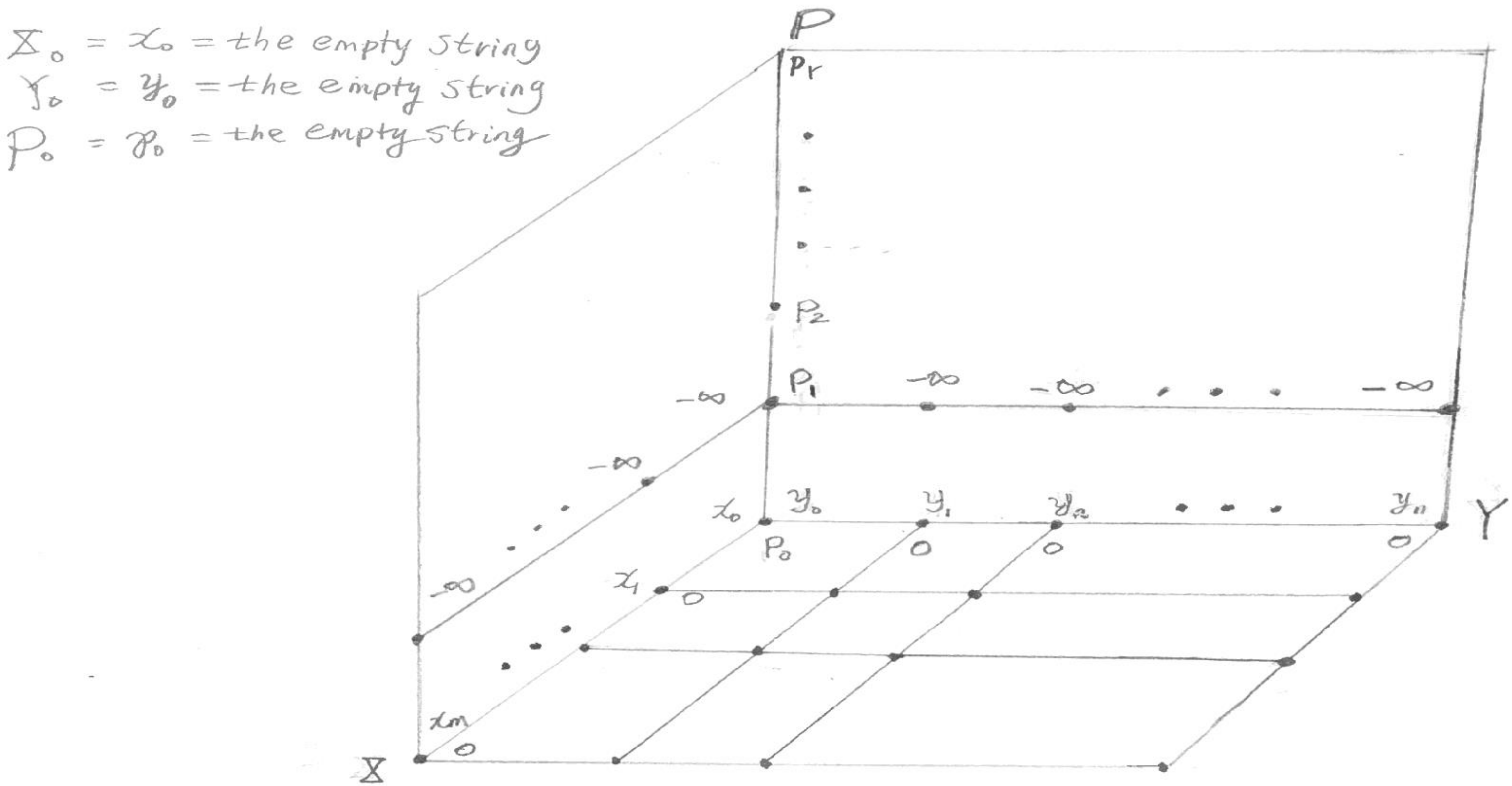


The Algorithm

-Filling in the boundary cells.

[5] If k = 0 or P_k is an empty string. Then CLCSSeqStr(X, Y; P_k) = LCSSeqStr(X, Y) in [1]. The cells of M[i][j][0], where $1 \leq i \leq m$ and $1 \leq j \leq n$, can be filled in by the following rules.

If $x_i = y_j$, then M[i][j][0] = M[i - 1][j - 1][0] + 1. If $x_i \neq y_j$, then M[i][j][0] = M[i - 1][j][0].



The Algorithm

-Filling in other cells.

-**Claim 1.** Suppose that $X_i = x_1\ x_2\ ...\ x_i$, $Y_j = y_1\ y_2\ ...\ y_j$, and $P_k = p_1\ p_2\ ...\ p_k$, where $1 \leq i \leq m$, $1 \leq j \leq n$, $1 \leq k \leq r$. If $Z[i, j, k] = z_1\ z_2\ ...\ z_a$ is a string satisfying conditions

(1) it is a subsequence of $X_i = x_1\ x_2\ ...\ x_i$,

(2) it is a suffix of $Y_j = y_1\ y_2\ ...\ y_j$,

(3) it has $P_k = p_1\ p_2\ ...\ p_k$ as a subsequence,

(4) under (1), (2) and (3), its length is as large as possible,

where $1 \leq i \leq m$, $1 \leq j \leq n$, and $1 \leq k \leq r$.

Then we have only the following possible cases and the statement in each case is true.

The Algorithm

-Filling in other cells.

-<u>Claim 1.</u>

Case 1. $x_i = y_j = p_k$. We have $|Z[i, j, k]| = |Z[i - 1, j - 1, k - 1]| + 1$ in this case.

Case 2. $x_i = y_j \neq p_k$. We have $|Z[i, j, k]| = |Z[i - 1, j - 1, k]| + 1$ in this case.

Case 3. $x_i \neq y_j$, $x_i \neq p_k$, and $y_j = p_k$. We have $|Z[i, j, k]| = |Z[i - 1, j, k]|$ in this case.

Case 4. $x_i \neq y_j$, $x_i \neq p_k$, and $y_j \neq p_k$. We have $|Z[i, j, k]| = |Z[i - 1, j, k]|$ in this case.

Case 5. $x_i \neq y_j$, $x_i = p_k$, and $y_j \neq p_k$. This case cannot happen.

The Algorithm

-Filling in other cells.

-**<u>Claim 2.</u>** Suppose there is not a string which is a subsequence of $X_i = x_1\ x_2\ ...\ x_i$,

a suffice of $Y_j = y_1\ y_2\ ...\ y_j$, and has $P_k = p_1\ p_2\ ...\ p_k$, as a subsequence, where

$1 \leq\ i \leq\ m$, $1 \leq\ j \leq\ n$, $1 \leq k \leq r$. Namely, Z[i, j, k] doesn't exist. Then

[1]. If $x_i = y_j = p_k$, then there is not a string which is a subsequence of X_{i-1}

$= x_1\ x_2\ ...\ x_{i-1}$, a suffix of $Y_{j-1} = y_1\ y_2\ ...\ y_{j-1}$, and has $P_{k-1} = p_1\ p_2\ ...\ p_{k-1}$

as a subsequence. **Namely, if Z[i, j, k] does not exist, then Z[i - 1, j - 1, k - 1] does not exist either.**

The Algorithm

-Filling in other cells.

-<u>Claim 2.</u>

[2]. If $x_i = y_j \neq p_k$, then there is not a string which is a subsequence of

$X_{i-1} = x_1\ x_2\ ...\ x_{i-1}$, a suffix of $Y_{j-1} = y_1\ y_2\ ...\ y_{j-1}$, and has $P_k = p_1\ p_2\ ...\ p_k$

as a subsequence. **Namely, if Z[i, j, k] does not exist, then Z[i - 1, j - 1, k]**

does not exist either.

The Algorithm

-Filling in other cells.

-<u>Claim 2.</u>

[3]. If $x_i \neq y_j$, $x_i \neq p_k$, and $y_j = p_k$, then there is not a string which is a subsequence of $X_{i-1} = x_1\ x_2 \ldots x_{i-1}$, a suffix of $Y_j = y_1\ y_2 \ldots y_j$, and has $P_k = p_1\ p_2 \ldots p_k$ as a subsequence. **Namely, if Z[i, j, k] does not exist, then Z[i - 1, j, k] does not exist either.**

The Algorithm

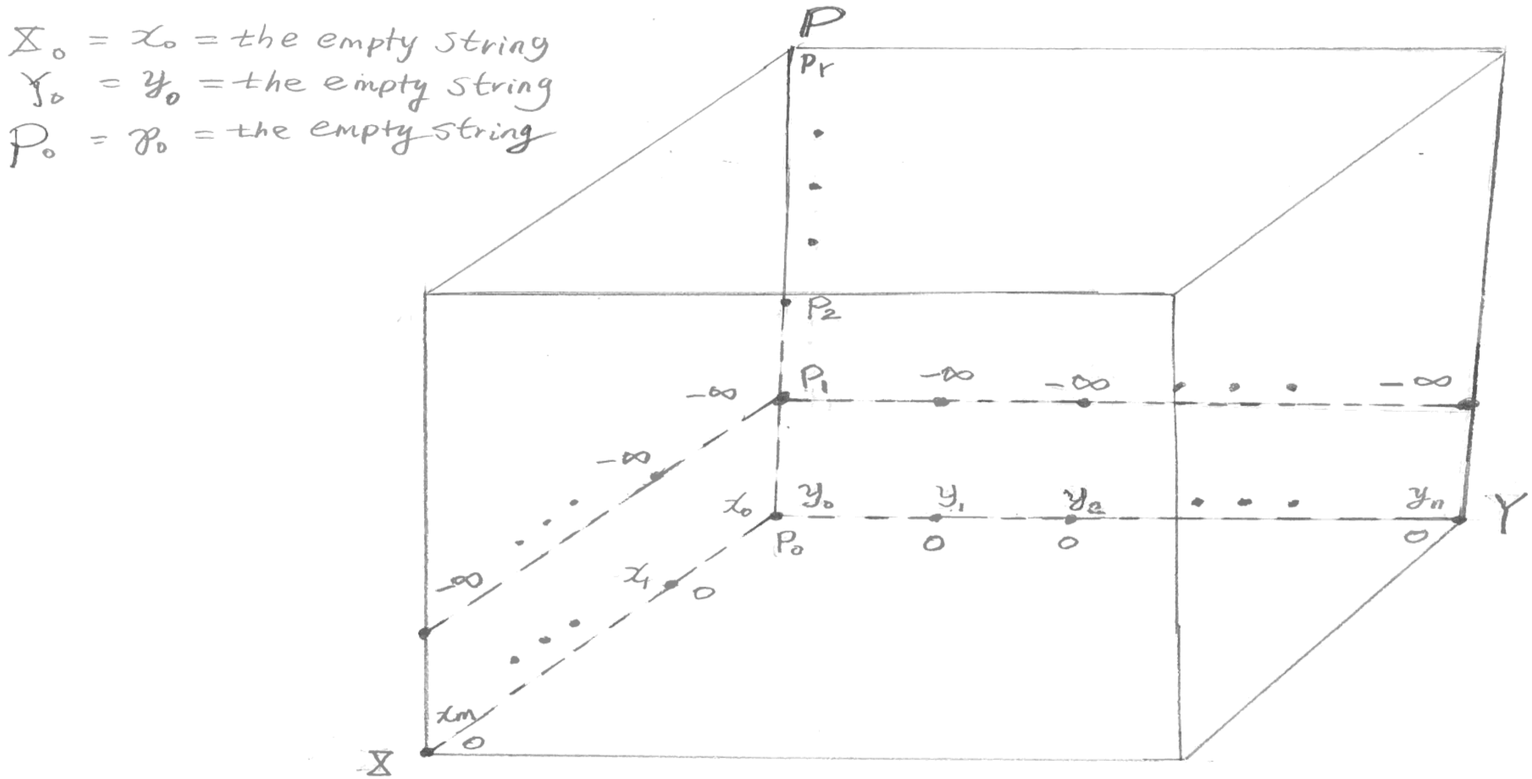
-Filling in other cells.

-<u>Claim 2.</u>

[4]. If $x_i \neq y_j$, $x_i \neq p_k$, and $y_j \neq p_k$, then there is not a string which is a subsequence for $X_{i-1} = x_1\ x_2\ ...\ x_{i-1}$, a suffix of $Y_j = y_1\ y_2\ ...\ y_j$, and has $P_k = p_1\ p_2\ ...\ p_k$ as a subsequence. **Namely, if Z[i, j, k] does not exist, then Z[i - 1, j, k] does not exist either.**

The Algorithm

-**Claim 3.** Let U_k be a longest string which is a subsequence of X, a substring of Y, and has P_k as a subsequence. Then $|U_k| = \max\{|Z[i, j, k]| : 1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq r\}$.
Thus $|U_r| = \max\{|Z[i, j, r]| : 1 \leq i \leq m, 1 \leq j \leq n\} = |CLCSSubStr(X, Y; P)|$.



The Algorithm

-$|CLCSSeqStr(X, Y; P)| = \max\{M[i][j][r]: 0 \le i \le m, 0 \le j \le n\}$.

-We can also find the CLCSSeqStr(X, Y; P) when we write a program.

-The time complexity of our algorithm is

$$O((|X| + 1)(|Y| + 1)(|P| + 1)) \sim O(|X|\ |Y|\ |P|).$$

-The space complexity of our algorithm also is

$$O((|X| + 1)(|Y| + 1)(|P| + 1)) \sim O(|X|\ |Y|\ |P|).$$

Thanks